Girraween High School

2018 Year 12 Mathematics Half Yearly Examination

General Instructions

- Working Time - 1 hour & 30 minutes
- Calculators and ruler may be used
- All necessary working out must be shown
- Write on one side of the paper only

Total Marks - 98

- Attempt all questions
- Marks may be deducted for careless or badly arranged work
- Start each question on a new sheet of paper

Question 1 (1 mark)

What is the value of $\frac{\pi^2}{6}$ correct to 3 significant figures?

A. 1.64 B. 1.65 C. 1.644 D. 1.645

Question 2 (1 mark)

Which one of the following statements is true for the equation $7x^2 - 5x + 2 = 0$

A. It has no real roots

B. It has one real root

C. It has two distinct roots

D. It has three real roots

Question 3 (1 mark)

For what values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

A. $x < -\frac{1}{6}$ B. $x > -\frac{1}{6}$ C. $x < -6$ D. $x > 6$

Question 4 (1 mark)

What is the value of $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$?

A. 2 B. 1 C. undefined D. 0

Question 5 (1 mark)

The value of $\sum_{k=3}^{10} 2k + 1$ is

A. 120 B. 91 C. 122 D. 112

Question 6 (1 mark)

The quadratic equation $x^2 + 3x - 1 = 0$ has roots α and β.

What is the value of $\alpha\beta + (\alpha + \beta)$?

A. 4 B. 2 C. -4 D. -2

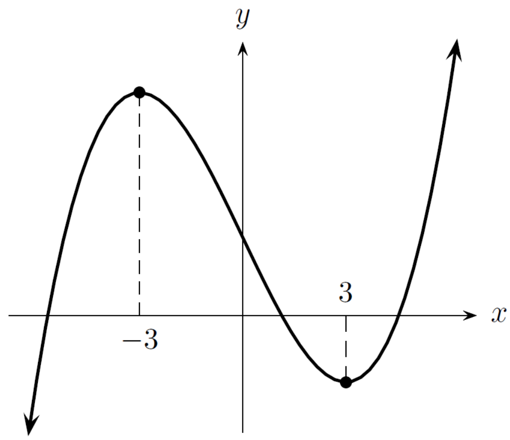
Question 7 (1 mark)

The angle of inclination that the line $3x + 2y - 7 = 0$ makes with the positive x-axis is closest to:

A. $56°$ B. $124°$ C. $34°$ D. $146°$

Question 8 (1 mark)

For the graph of $y = f(x)$, for what x values is $f'(x)$ negative?



A. $x < -3$ and $x > 3$ B. $-3 < x < 3$ C. $x \leq -3$ and $x \geq 3$ D. $-3 \leq x \leq 3$

Question 9 (1 mark)

The domain of the function $f(x) = \dfrac{1}{\sqrt{4x^2 - 1}}$ is:

A. $-\dfrac{1}{2} < x < \dfrac{1}{2}$

B. $x < -\dfrac{1}{2}$ and $x > \dfrac{1}{2}$

C. $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$

D. $x \leq -\dfrac{1}{2}$ and $x \geq \dfrac{1}{2}$

Question 10 (1 mark)

If y is decreasing at an increasing rate, which of the following is true?

A. $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} < 0$

B. $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$

C. $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$

D. $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} > 0$

Question 11 on the next page

Question 11 (20 marks)

(a) Find the derivative of the following:

i. $y = e^{2x+1} - e^{-\frac{1}{2}x}$ [2]

ii. $y = \dfrac{1}{1+2e^x}$ [2]

iii. $y = \dfrac{x^2+1}{x-1}$ [2]

iv. $y = 3x(2x+1)^3$ [2]

v. $y = \dfrac{2+e^x}{e^{2x}}$ [2]

(b) Find the primitive of the following:

i. $\dfrac{e^{2x}}{2}$ [2]

ii. $\sqrt{2x-1}$ [2]

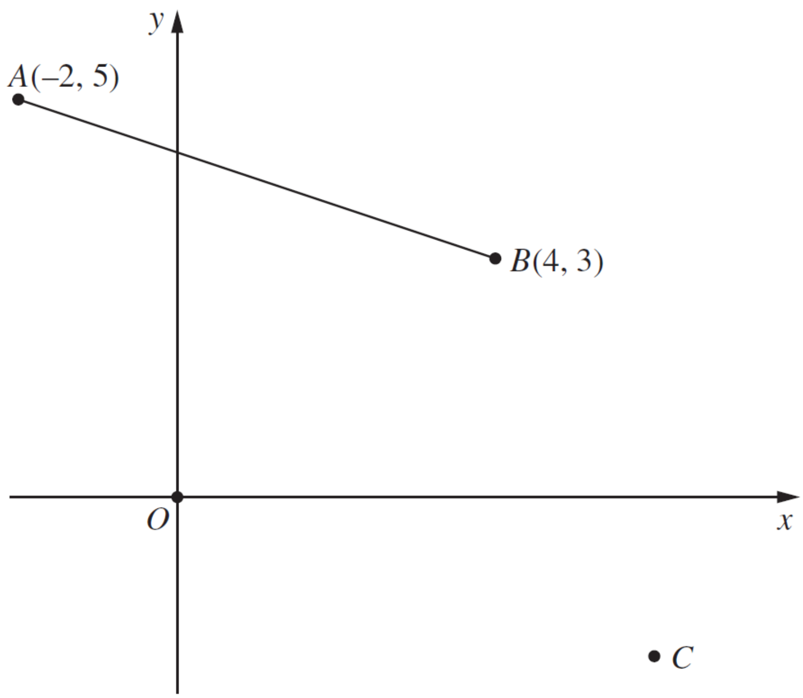
(c) Find the values of a and b such that $\dfrac{1}{4-\sqrt{13}} = a+b\sqrt{13}$, where a and b are rational numbers. [3]

(d) Prove that $\sin\theta + \cot\theta\cos\theta = \text{cosec}\theta$ [3]

The exam continues on the next page

Question 12 (19 marks)

(a) The diagram below shows the points $A(-2,5)$, $B(4,3)$ and $O(0,0)$. The point C is the fourth vertex of the parallelogram $OABC$.



i. Show that the equation of AB is $x + 3y - 13 = 0$. [2]

ii. Show that the length of AB is $2\sqrt{10}$. [1]

iii. Calculate the perpendicular distance from O to the line AB. [1]

iv. Calculate the area of parallelogram $OABC$. [1]

v. Hence or otherwise find the perpendicular distance from O to the line BC. [2]

(b) The second term of arithmetic progression is 7 and the seventh term is 52

i. Find the common difference. [2]

ii. Find the value of the first term which is greater than 1000 [2]

iii. Find the sum of the first ten terms. [1]

(c) The equation $(x-1)^2 = -4y + 16$ represents a parabola.

i. Find the coordinates of the vertex. [2]

ii. Find the focal length and the equation of directrix. [2]

iii. Sketch parabola, clearly showing the directrix, the focus and the x-intercept. [3]

The exam continues on the next page

Question 13 (18 marks)

(a) Michael buys five tickets in a raffle in which 20 tickets are sold. Three different tickets are to be drawn out without replacement for first, second and third prizes. Find the probability that:

i. Michael wins all three prizes. [2]

ii. Michael does not win a prize. [2]

iii. Michael wins at least one prize. [1]

iv. Michael wins exactly one prize. [2]

(b) i. Find $\dfrac{d}{dx}\, e^{1-x^2}$ [1]

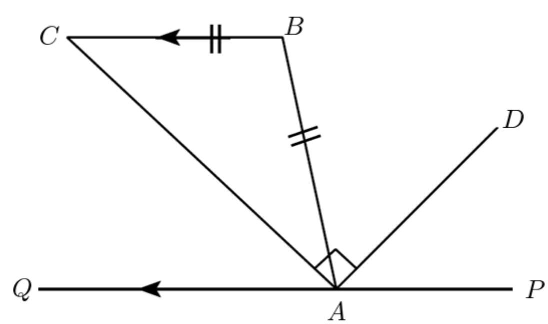
ii. Hence or otherwise find $\displaystyle\int xe^{1-x^2}\, dx$ [2]

(c) i. Use the trapezoidal rule with 5 function values to find an approximation to [3]

$$\int_1^5 \frac{1}{x}\, dx.$$

ii. State whether the approximation found in part (i) is greater or less than the exact value of $\displaystyle\int_1^5 \frac{1}{x}\, dx$. Justify your answer. [1]

(d) In the diagram below ABC is a triangle with $AB = BC$. The line PQ passes through A parallel to BC, and the line AD is perpendicular to AC.



i. Let $\angle BAC = \theta$ and prove that AC bisects $\angle QAB$. [2]

ii. Prove that AD bisects $\angle PAB$. [2]

The exam continues on the next page

Question 14 (17 marks)

(a) Consider the function $f(x) = xe^x$

i. Find the coordinates of the stationary points and determine their nature. [3]

ii. Find the coordinates of the points of inflexion. [2]

iii. State the behaviour of $y = f(x)$ as $x \to \infty$ and as $x \to -\infty$. [2]

iv. Sketch the graph of the curve $y = f(x)$, showing intercepts, stationary points and point of inflexion. [3]

(b) Let A and B be fixed points $(-1, 0)$ and $(1, 2)$. Let P be the variable point (x, y) such that $\angle APB = 90°$. Find the equation of the locus of P. [3]

(c) Consider the series $1 + x + (1 + x)^2 + (1 + x)^3 + \cdots$.

i. Show that the series is geometric. [1]

ii. Find the values of x for which this series has a limiting sum. [2]

iii. Find the limiting sum of this series in terms of x. [1]

The exam continues on the next page

Question 15 (14 marks)

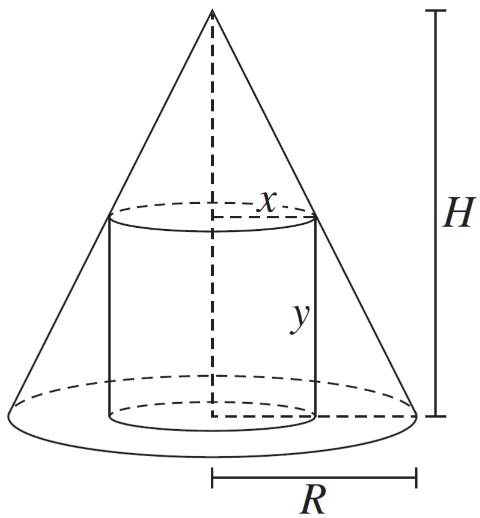
(a) Find the volume of the solid generated by rotating the area enclosed by the curve $y = x^2 + 1$, $x = 1$ and the coordinate axes around the y-axis. [3]

(b) The graph of $y = e^{kx}$ has a tangent at $x = a$ that passes through the origin, where k and a are constants.

i. Show that $ka = 1$. [3]

ii. Find the area bounded by $y = e^{x/2}$ and its tangent at $x = 2$ and the y-axis. [3]

(c) The diagram below shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants.



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

i. Show that the volume, V, of the cylinder can be written as [2]

$$V = \frac{H}{R}\pi x^2 (R - x)$$

ii. Find the value of x that maximises V. [3]

End of exam

2018 Yr 12 2u Half Yearly

Q1/ (A)

Q2/ $\Delta = 25 - 4(7)(2) < 0$

$\therefore$ (A)

Q3/ $f(x) = 2x^3 + x^2$

$f'(x) = 6x^2 + 2x$

$f''(x) = 12x + 2.$

$12x + 2 < 0$

$12x < -2$

$x < -\frac{1}{6}.$ $\therefore$ (A)

Q4/ $\lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = 2.$ $\therefore$ (A)

Q5/

$a = 7$ $d = 2$ $n = 8$

$S_n = \frac{8}{2}(7 + 21) = 112$ $\therefore$ (D)

Q6/

$\alpha\beta = -1$ $\alpha + \beta = -3$ $\therefore$ (C)

Q7/

$3x + 2y - 7 = 0$

$2y = -3x + 7$

$y = -\frac{3}{2}x + \frac{7}{2}.$

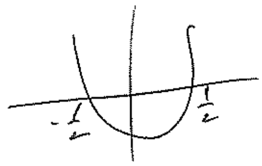
$\tan\alpha = -\frac{3}{2}$ $\alpha = 123°41'$.

$\therefore$ (B)

Q8/ (B)

Q9/ $4x^2 - 1 > 0.$

$(2x-1)(2x+1) > 0.$



$\therefore x > \frac{1}{2}$

& $x < -\frac{1}{2}.$

$\therefore$ (B).

Q10/ (C) or (A)

Q11/

(a)

i $y = e^{2x+1} - e^{-\frac{1}{2}x}.$

$y' = 2e^{2x+1} + \frac{1}{2}e^{-\frac{1}{2}x}.$

ii $y = (1 + 2e^x)^{-1}$

$y' = -(1 + 2e^x)^{-2} \times 2e^x$

$y' = \frac{-2e^x}{(1 + 2e^x)^2}$

iii $u = x^2 + 1$ $v = x - 1$ $y' = \frac{2x(x-1) - (x^2+1)}{(x-1)^2}$

$u' = 2x$ $v' = 1$

$y' = \frac{x^2 - 2x - 1}{(x-1)^2}$

iv. $u = 3x$ $v = (2x+1)^3$

$u' = 3$ $v' = 3(2x+1)^2 \times 2.$

$v' = 6(2x+1)^2$

$y' = 3(2x+1)^3 + 18x(2x+1)^2$

Q11

(a)

i. $y = 2e^{-2x} + e^{-x}$

$y' = -4e^{-2x} - e^{-x}$.

(b)

(i) $\frac{1}{2}\int e^{x/2}\,dx$.

$= e^{x/2} + c$

(ii) $\int (2x-1)^{1/2}\,dx$

$= \frac{(2x-1)^{3/2}}{2 \times 3/2} + c$.

$= \frac{1}{3}(2x-1)^{3/2} + c$.

(c) $\frac{1}{4-\sqrt{13}} \times \frac{4+\sqrt{13}}{4+\sqrt{13}}$

$= \frac{4+\sqrt{13}}{16-13} = \frac{4}{3} + \frac{1}{3}\sqrt{13}$

$\therefore a = \frac{4}{3}$ & $b = \frac{1}{3}$.

(d) $LHS = \sin\theta + \cot\theta\cos\theta$

$= \sin\theta + \frac{\cos^2\theta}{\sin\theta}$.

$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}$.

$= \frac{1}{\sin\theta} = \operatorname{cosec}\theta = RHS$.

Q12

(a)

(i) $m = \frac{5-3}{-2-4} = \frac{2}{-6} = -\frac{1}{3}$

$y - 3 = -\frac{1}{3}(x-4)$.

$3y - 9 = -x + 4$

$x + 3y - 13 = 0$.

(ii)

$D_{AB} = \sqrt{6^2 + (-2)^2}$

$= \sqrt{40} = 2\sqrt{10}$

(iii) $d = \frac{|-13|}{\sqrt{1^2+3^2}} = \frac{13}{\sqrt{10}}$.

(iv) $A = 2\sqrt{10} \times \frac{13}{\sqrt{10}} = 26u^2$

(v). $D_{OA} = \sqrt{2^2 + 5^2}$

$= \sqrt{29}$.

$\therefore \sqrt{29} \times d = 26$

$\therefore d = \frac{26}{\sqrt{29}}$.

(b)

(i) $a + d = 7$

$a + 6d = 52$

$\therefore 5d = 45 \quad \therefore d = 9$.

Q12

(b)

(ii) $a = -2$

$a + (n-1)d > 1000$

$-2 + (n-1) \times 9 > 1000$

$-2 + 9(n-1) > 1000$

$9n - 9 > 1002.$

$9n > 1011$

$n > \frac{1011}{9} \quad \therefore n = 113$

$T_{113} = -2 + 112 \times 9 = 1006.$

(iii) $S_{10} = \frac{10}{2}\left(2(-2) + 9 \times 9\right)$

$= 385.$

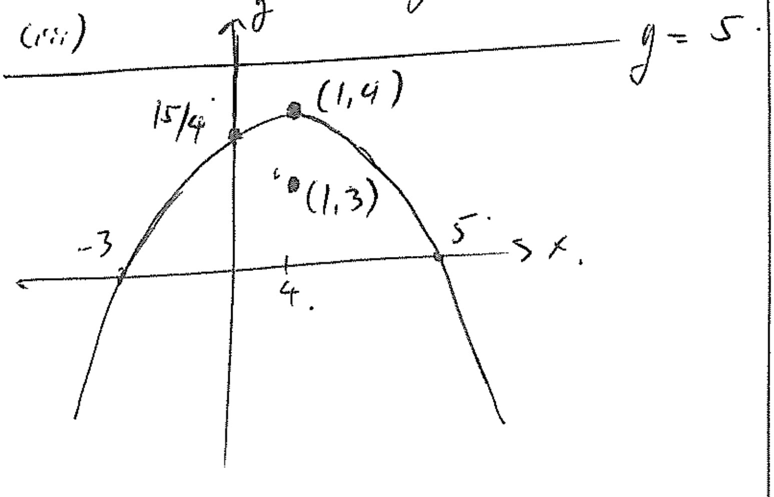
(c)

(i) $(x-1)^2 = -4(y-4).$

$V = (1, 4).$

(ii) $a = 1 \quad D: y = 5.$

(iii)



Q13

(a) $\frac{5}{20} \times \frac{4}{19} \times \frac{3}{18} = \frac{1}{114}$

(ii) $\frac{15}{20} \times \frac{14}{19} \times \frac{13}{18} = \frac{91}{228}$

(iii) $1 - \frac{91}{228} = \frac{137}{228}$

(iv) $3 \times \frac{5}{20} \times \frac{15}{19} \times \frac{14}{18} = \frac{35}{76}.$

(b)

(i) $\frac{d}{dx} e^{1-x^2} = -2x e^{1-x^2}$

(ii) $\int -2x e^{1-x^2} dx = e^{1-x^2}$

$\therefore \int x e^{1-x^2} dx = -\frac{1}{2} e^{1-x^2} + C.$

(c)

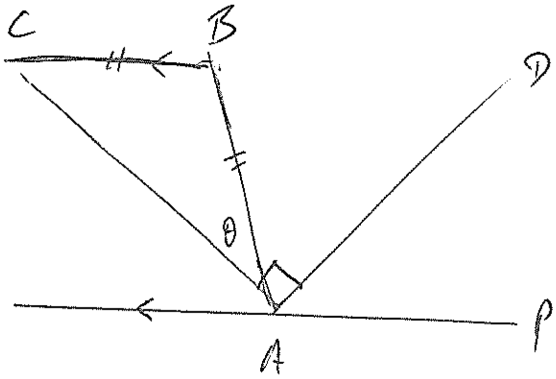
(i) $h = \frac{5-1}{4} = 1.$

$A \approx \frac{1}{2}\left[\frac{1}{1} + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right) + \left(\frac{1}{5}\right)\right]$

$\approx \frac{101}{60}$

(ii) Greater, as $y = \frac{1}{x}$ is concave up for $1 \leq x \leq 5.$

Q13

(d) 

(i) $\angle BCA = \theta$ (equal $\angle$ opposite equal sides)

$\angle CAQ = \theta$ (alternate $\angle$'s, $CB \parallel QA$).

$\therefore$ AC bisects $\angle QAB$

(ii) $\theta + 90 + \angle DAP = 180$ (straight $\angle$)

$\therefore \angle DAP = 90 - \theta$.

$\angle DAB = 90 - \theta$. (adjacent $\angle$)

$\therefore$ DA bisects $\angle BAP$.

Q14

(c) $u = x \quad v = e^x$

(i) $u' = 1 \quad v' = e^x$

$f'(x) = e^x + xe^x$.

$f'(x) = e^x(1+x)$.

$f'(x) = 0$ when $x = -1$.

$f(-1) = -\frac{1}{e}$ $\therefore$ stat pt at $(-1, -\frac{1}{e})$

$f''(x) = e^x + e^x + xe^x$

$= 2e^x + xe^x$.

$f''(-1) = 2e^{-1} - e^{-1}$

$= e^{-1} > 0$

$\therefore$ min at $(-1, -\frac{1}{e})$

(ii) $f''(x) = 0$ when $e^x(2 + x) = 0$.

$\therefore x = -2$.

$f(-2) = -2e^{-2} = -\frac{2}{e^2}$

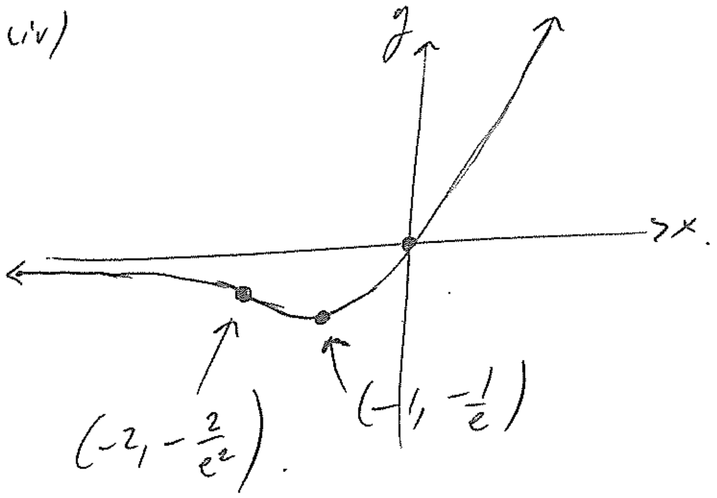
possible point of inflexion at $(-2, -\frac{2}{e^2})$.

x	-3	-2	-1
y''	$-\frac{1}{e^3}$	0	$\frac{1}{e}$

$\therefore (-2, -\frac{2}{e^2})$ is a point of inflexion.

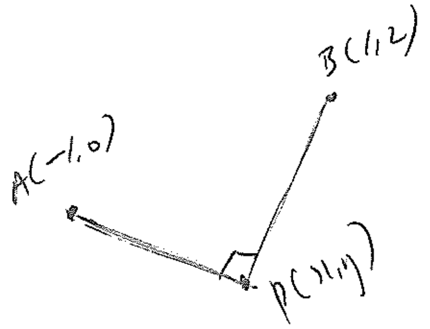
(iii) $x \to \infty \quad y \to \infty$

$x \to -\infty \quad y \to 0$.

(iv) 

Q14

(b)



$$\frac{y}{x+1} \times \frac{y-2}{x-1} = -1.$$

$$y(y-2) = -(x^2-1)$$

$$y^2-2y = -x^2+1$$

$$x^2+y^2-2y-1=0.$$

(c)

(i) $$\frac{T_2}{T_1} = \frac{(1+x)^2}{(1+x)} = 1+x$$

$$\frac{T_3}{T_2} = \frac{(1+x)^3}{(1+x)^2} = 1+x$$

$\therefore$ Series is geometric.

(ii) Need $-1 < 1+x < 1$

$\therefore -2 < x < 0$

(iii)

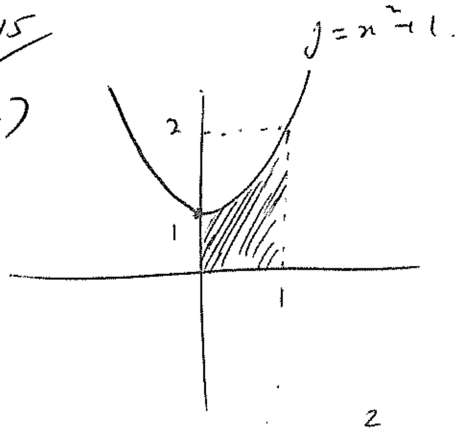
$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1+x}{1-(1+x)} = \frac{1+x}{-x}$$

$$= \frac{-1-x}{x}.$$

Q15

(a)



$$V = \pi(1)^2 \times 2 - \pi\int_1^2 y-1\,dy.$$

$$V = 2\pi - \pi\left[\tfrac{y^2}{2} - y\right]_1^2$$

$$V = 2\pi - \pi\left(2-2-\left(\tfrac{1}{2}-1\right)\right).$$

$$V = 2\pi - \frac{\pi}{2} = \frac{3}{2}\pi$$

(b)(i) $y = e^{kx}$

$y' = ke^{kx}$

$y(a) = e^{ka}$ $\quad y'(a) = ke^{ka}$

(a, e^{ka})

$$y - e^{ka} = ke^{ka}(x-a)$$

$$y - e^{ka} = ke^{ka}x - ake^{ka}.$$

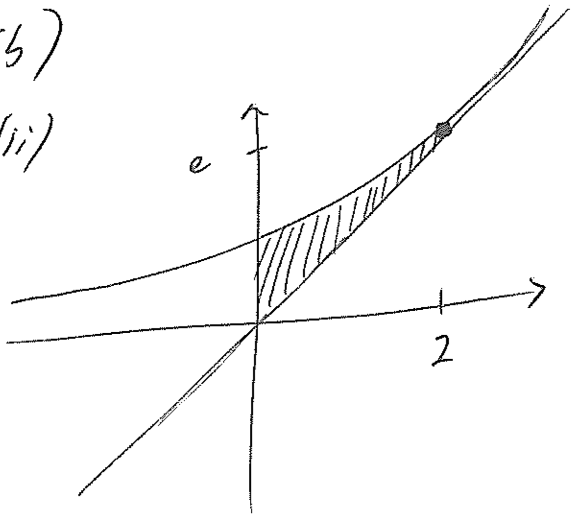
$$y = ke^{ka}x + (1-ak)e^{ka}$$

$\therefore 1-ak=0 \quad \therefore ak=1.$

Q15

(b)

(ii)



$$A = \int_0^2 e^{x/2}\,dx - \frac{1}{2}(2)(e)$$

$$= 2\left[e^{x/2}\right]_0^2 - e$$

$$= 2(e-1) - e$$

$$= 2e - 2 - e = e - 2.$$

(c)

(i) $\dfrac{x}{R} = \dfrac{H-y}{H}$ (ratio of matching sides of ||| △'s).

$\therefore Hx = R(H-y)$.

$Hx = RH - Ry$.

$Ry = RH - Hx$.

$y = \dfrac{H}{R}(R-x)$.

$V = \pi x^2 y$

$\therefore V = \pi x^2 \times \dfrac{H}{R}(R-x)$.

$V = \dfrac{H}{R}\pi x^2(R-x)$

(ii) $V = H\pi x^2 - \dfrac{H}{R}\pi x^3$

$V' = 2H\pi x - \dfrac{3H}{R}\pi x^2$.

$V' = H\pi x\left(2 - \dfrac{3}{R}x\right)$.

$\therefore V' = 0$ when $x = 0$

or $2 - \dfrac{3x}{R} = 0$.

$2 = \dfrac{3x}{R}$

$\therefore x = \dfrac{2R}{3}$.

$V'' = 2H\pi - \dfrac{6H}{R}\pi x$.

$V''\left(\dfrac{2R}{3}\right) = 2H\pi - \dfrac{6H}{R}\pi \times \dfrac{2R}{3}$.

$= 2H\pi - 4H\pi < 0$

$\therefore x = \dfrac{2R}{3}$ gives max V.